

Example 2

Find $\frac{dy}{dx}$ in each case.

a $y = (5x + 1)^4$
$$\frac{dy}{dx} = 4(5x + 1)^3 \times \frac{d}{dx}(5x + 1)$$
$$= 4(5x + 1)^3 \times 5$$
$$= 20(5x + 1)^3$$

c $y = \sqrt{2x + 3}$
$$= (2x + 3)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(2x + 3)^{-\frac{1}{2}} \times \frac{d}{dx}(2x + 3)$$
$$= \frac{1}{2}(2x + 3)^{-\frac{1}{2}} \times 2$$
$$= (2x + 3)^{-\frac{1}{2}}$$

b $y = 3(1 + x^2)^7$
$$\frac{dy}{dx} = 3 \times 7(1 + x^2)^6 \times \frac{d}{dx}(1 + x^2)$$
$$= 21(1 + x^2)^6 \times 2x$$
$$= 42x(1 + x^2)^6$$

d $y = \frac{1}{(3x^3 + 1)^2} = (3x^3 + 1)^{-2}$
$$\frac{dy}{dx} = -2(3x^3 + 1)^{-3} \times \frac{d}{dx}(3x^3 + 1)$$
$$= -2(3x^3 + 1)^{-3} \times 9x^2$$
$$= \frac{-18x^2}{(3x^3 + 1)^3}$$

EXERCISE 1

1 Find $\frac{dy}{dx}$.

a $y = (2x + 5)^3$, let $u = 2x + 5$.

b $y = (x^2 + 7)^4$, let $u = x^2 + 7$.

2 Find $\frac{dy}{dx}$.

a $y = (3x - 4)^4$

b $y = (8x + 11)^5$

c $y = (x^2 - 3)^2$

d $y = (3x^3 + 1)^3$

e $y = (1 - 3x)^4$

f $y = (3 - x)^{10}$

3 Differentiate the following.

a $y = (1 + 3x)^{-1}$

b $y = \frac{1}{(2x + 1)^2}$

c $y = \frac{1}{(5x + 2)^3}$

d $y = \frac{3}{(4x - 1)^2}$

e $y = \frac{5}{(x^2 + 1)}$

f $y = \frac{10}{(1 - x)^4}$

4 a If $f(x) = \sqrt{2x^2 - 1}$, show that $f'(x) = \frac{2x}{\sqrt{2x^2 - 1}}$.

b Find $f'(2)$.

5 Find $f'(x)$.

a $f(x) = \sqrt{5x^2 + 3}$

b $f(x) = (3x + 1)^{\frac{1}{3}}$

c $f(x) = \frac{1}{\sqrt{x^2 - 3}}$

d $f(x) = \frac{1}{x^3 + 1}$

e $f(x) = \frac{1}{\sqrt{8x + 7}}$

f $f(x) = \frac{1}{\sqrt{x + 2}}$

6 Find the equation of the tangent to the curve $y = (x^2 + 1)^4$ at the point $(1, 16)$.

- 7 Find the equation of the normal to the curve $y = (3x + 1)^{\frac{2}{3}}$ at the point where $x = 5$.
- 8 For the curve $y = (2 + x^2)^3$, find the coordinates of the point where the gradient is zero.
- 9 Given $y = (3x + 1)^2 + (3x + 1)^3$, find the value of $\frac{dy}{dx}$ at $x = \frac{1}{3}$.
- 10 Find the equation of the tangent to the curve $y = \sqrt{x^2 + 3}$ at the point $(1, 2)$.
- 11 Differentiate the following
- a $y = (2\sqrt{x} - 3x)^3$ b $y = \sqrt{1 - \frac{1}{x}}$ c $y = \frac{2}{x^{\frac{3}{2}} + 2}$
- d $y = \left(\frac{1}{1 + \sqrt{x}}\right)^2$ e $y = \sqrt{x^2 - \frac{1}{x}}$ f $y = 10x^2 + \sqrt{2x - 3}$
- 12 a Find the coordinates of the two stationary points of $y = \frac{8}{(x + 1)} + 2x + 1$.
b Show that one is a local maximum and the other is a local minimum.
- 13 a Find the coordinates of the stationary point of $y = \frac{32}{(3x + 1)^2} + 3x - 3$.
b Show that this is a local minimum.

4.2 Product Rule

Consider the product uv where u and v are functions of x .

$$y = uv \quad [1]$$

Let x increase by a small amount δx and let the corresponding increase in u be δu , in v be δv and in y be δy .

$$\text{So } y + \delta y = (u + \delta u)(v + \delta v) \quad [2]$$

$$[2] - [1] \quad y + \delta y - y = (u + \delta u)(v + \delta v) - uv$$

$$\delta y = u \delta v + v \delta u + \delta u \delta v$$

$$\frac{\delta y}{\delta x} = u \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} + \delta u \frac{\delta v}{\delta x}$$

$$\text{As } \delta x \rightarrow 0 \quad \frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}, \quad \frac{\delta v}{\delta x} \rightarrow \frac{dv}{dx} \quad \text{and} \quad \frac{\delta u}{\delta x} \rightarrow \frac{du}{dx}.$$

And as $\delta x \rightarrow 0$ δu and δv both tend towards zero.

$$\boxed{\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}}$$

Remember this important result.

Example 1

Find $\frac{dy}{dx}$:

a $y = x(x+2)^2$

b $y = (x+3)^2(2x+1)^3$

c $y = (2x+1)^2 \sqrt{4x-3}$

a We have $y = uv$, where $u = x$ and $v = (x+2)^2$

$$\left[\therefore \frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = 2(x+2) \right]$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= x \times 2(x+2)^1 + (x+2)^2 \times 1 \\ &= (x+2)(2x+x+2) \\ &= (x+2)(3x+2) \end{aligned}$$

b $y = (x+3)^2(2x+1)^3$

We have $y = uv$, where $u = (x+3)^2$ and $v = (2x+1)^3$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= (x+3)^2 \times 3(2x+1)^2 \times 2 + (2x+1)^3 \times 2(x+3)^1 \\ &= (x+3)(2x+1)^2 [6(x+3) + (2x+1) \times 2] \\ &= (x+3)(2x+1)^2 (10x+20) \\ &= 10(x+3)(2x+1)^2 (x+2) \end{aligned}$$

c $y = (2x+1)^2 \sqrt{4x-3}$

We have $y = uv$, where $u = (2x+1)^2$ and $v = (4x-3)^{\frac{1}{2}}$

$$\begin{aligned} \frac{dy}{dx} &= (2x+1)^2 \times \frac{1}{2}(4x-3)^{-\frac{1}{2}} \times 4 + (4x-3)^{\frac{1}{2}} \times 2(2x+1)^1 \times 2 \\ &= \frac{(2x+1)}{(4x-3)^{\frac{1}{2}}} [2(2x+1) + (4x-3) \times 4] \\ &= \frac{2x+1}{\sqrt{4x-3}} [20x-10] \\ &= \frac{10(2x+1)(2x-1)}{\sqrt{4x-3}} \end{aligned}$$

EXERCISE 2

Differentiate the following with respect to x and simplify your answers.

1 $y = x(2x+1)^2$

2 $y = x^2(3x-1)^2$

3 $y = x^3(x-1)^2$

4 $y = (x+2)^2(x+3)^2$

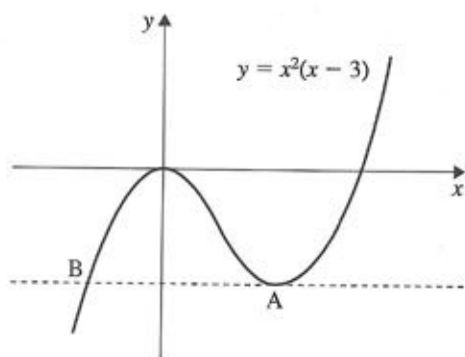
5 $y = 2x(1-x)^3$

6 $y = (4x^3+1)(x-1)^2$

7 $y = 5x(1+2x)^4$

8 $y = (3x+1)^4(x-3)$

- 9 $y = (x + 2)x^{\frac{3}{2}}$
- 10 $y = (4x + 1)^2 \sqrt{x}$
- 11 $y = x^2 \sqrt{2x + 1}$
- 12 $y = x^3 \sqrt{4x - 1}$
- 13 Find the equation of the tangent to the curve $y = 6x \sqrt{x + 8}$ at the point (1, 18).
- 14 Find the gradient of the curve $y = x^3(x + 1)^2$ at the point (1, 4).
- 15 Find the x coordinates of the two points where the gradient of the curve $y = \sqrt{x}(x - 2)^2$ is zero.
- 16 a Find the x coordinates of the turning points of the curve $y = (x + 3)^2(x - 1)$.
b Determine the nature of each turning point.
- 17 a For the curve $y = (x + 2)(x - 4)^2$, find
i the points where it meets the axes,
ii the turning points.
b Sketch the curve.
- 18 The curve $y = x^2(x - 3)$ has a minimum value at the point A.



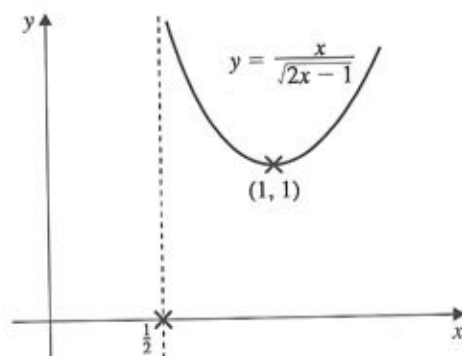
- a Find the coordinates of A.
- b The tangent to the curve at A meets the curve again at B. Find the coordinates of B.
- 19 i Given that $y = (x + 1)\sqrt{2x - 1}$, show that $\frac{dy}{dx}$ can be written in the form $\frac{kx}{\sqrt{2x - 1}}$ and state the value of k .
ii Hence evaluate

$$\int_1^5 \frac{x}{\sqrt{2x - 1}} dx$$

When $x = 1$, $\frac{d^2y}{dx^2} = \frac{-1 + 2}{1} = 1$ which is greater than zero.

This tells us that there is a minimum value at $(1, 1)$.

Here is a sketch of the curve.



Notice that as $x \rightarrow \frac{1}{2}$, $y \rightarrow \infty$.

EXERCISE 3

Find $\frac{dy}{dx}$.

1 $y = \frac{2x}{x+1}$

2 $y = \frac{x+7}{2x-1}$

3 $y = \frac{x^2}{3x+1}$

4 $y = \frac{4x+5}{x^2+1}$

5 $y = \frac{(x+1)^2}{x^2}$

6 $y = \frac{2x+1}{(x+1)^3}$

7 $y = \frac{(3x+2)^2}{(4x+1)^3}$

8 $y = \frac{x^3}{(x-3)^4}$

9 $y = \frac{(3x-4)^3}{(2x+1)^2}$

10 $y = \frac{\sqrt{x+1}}{x}$

11 $y = \frac{x}{\sqrt{4x+3}}$

12 $y = \frac{3x+1}{\sqrt{2x-1}}$

13 Find the equation of the tangent to the curve $y = \frac{x^2}{(x+1)^3}$ at the point $(1, \frac{1}{8})$.

14 Find the equation of the tangent to the curve $y = \frac{x+1}{2x-1}$ at the point $(0, -1)$.

15 Find the equation of the normal to the curve $y = \frac{x+4}{x^2+1}$ at the point $(0, 4)$.

16 Find the x coordinate of the point on the curve $y = \frac{2x}{\sqrt{4x-3}}$ where the gradient is zero.

17 Find the coordinates of the turning points of $y = \frac{1-x}{x^2+3}$. Determine their nature.

18 Show that the gradient of the curve $y = \frac{x^3}{\sqrt{x^2 + 1}}$ is positive for all values of x except $x = 0$.

19 Show that if $y = \frac{x^2}{\sqrt{x + 1}}$ then $\frac{dy}{dx} = \frac{x(3x + 4)}{2(x + 1)\sqrt{x + 1}}$.

4.4 Differentiating e^x

In Part 3 of this book we stated that when we differentiate the function e^x we obtain e^x . It is not necessary at this level to understand a proof of this result but we indicate below how we can 'justify' the result by a numerical method.

At this stage remember: $\frac{d}{dx}(e^x) = e^x$

For a function, $f(x)$, $f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$.

So if $f(x) = a^x$ we see that

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \left(\frac{a^{x+h} - a^x}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{a^x \cdot a^h - a^x}{h} \right) \\ &= a^x \lim_{h \rightarrow 0} \left(\frac{a^h - 1}{h} \right). \end{aligned}$$

We require the value of a such that $f'(a^x) = a^x$.

We need $\lim_{h \rightarrow 0} \left(\frac{a^h - 1}{h} \right) = 1$.

Try $a = 2$, $h = 0.001$: $\frac{2^{0.001} - 1}{0.001} \approx 0.693$. So $a = 2$ is too small.

Try $a = 3$, $h = 0.001$: $\frac{3^{0.001} - 1}{0.001} \approx 1.099$. So $a = 3$ is too large.

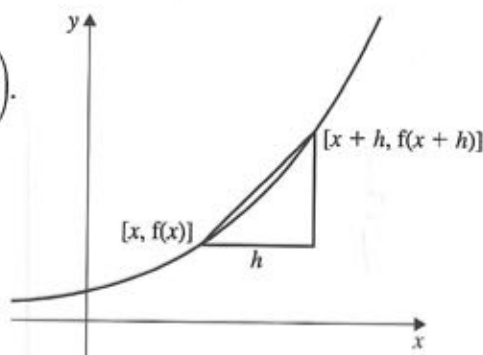
Try $a = 2.71$, $h = 0.001$: $\frac{2.71^{0.001} - 1}{0.001} \approx 0.997$. So $a = 2.71$ is too small.

Try $a = 2.72$, $h = 0.001$: $\frac{2.72^{0.001} - 1}{0.001} \approx 1.001$. So $a = 2.72$ is too large.

When this process is continued we obtain the number 2.718 281 828 correct to 9 decimal places. Note that this is an *approximate* value for e . The number e is in fact an irrational number which means that it cannot be written in the form $\frac{a}{b}$ where a and b are integers.

A frequently asked question is 'What is e ?'

Answer: e is just a *number* which has the unique property that $\frac{d}{dx}(e^x) = e^x$.

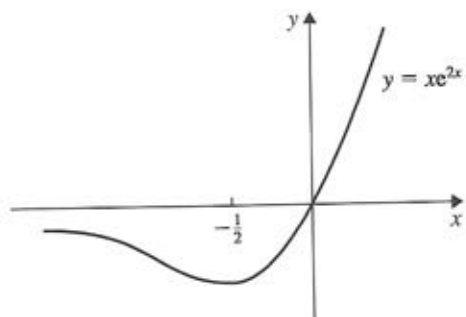


Example 1

Find the gradient of the curve $y = e^x$ at the point $(2, e^2)$.

$$\begin{aligned} y &= e^x \\ \frac{dy}{dx} &= e^x \end{aligned}$$

Here is a sketch of the curve.



EXERCISE 4

1 Find $\frac{dy}{dx}$ for each of the following.

a $y = e^x$

b $y = e^{3x}$

c $y = 2e^x$

d $y = e^{x^2}$

e $y = e^{2x+1}$

f $y = e^{-x}$

g $y = 5e^{4x}$

h $y = e^x + e^{2x}$

i $y = x^2 - e^{-x}$

j $y = \frac{1}{e^{2x}}$

k $y = \frac{4}{e^x}$

l $y = e^{x^3} + x^3$

2 Differentiate with respect to x .

a $x e^x$

b $x^2 e^x$

c $2x e^{2x}$

d $(x^2 + 1)e^x$

e $4x^3 e^{2x}$

f $x^2(e^{3x} - x)$

g $\frac{e^{3x}}{x}$

h $\frac{e^x + 1}{x^2}$

i $\frac{x^2}{e^x}$

j $e^x(1+x)^3$

k $e^{-x}(x^3 + 1)$

l $\frac{e^{2x}}{1 - e^x}$

3 Find the gradient of each curve at the given point.

a $y = x e^x$ at $x = 2$

b $y = x e^{x-2}$ at $(2, 2)$

c $y = e^x + x^2$ at $(0, 1)$

d $y = \frac{e^{2x} + 1}{e^{2x} - 1}$ at $x = 1$.

4 Find the equation of the tangent to the curve $y = e^x$ at the point where $x = 2$.

5 Find the equation of the tangent to the curve $y = x e^x$ at the point $(0, 0)$.

6 a Sketch the curves $y = e^x$ and $y = e^{x-2}$.

b Find the equation of the normal to the curve $y = e^{x-2}$ at the point where the curve meets the y -axis.

c Find the coordinates of the point where the normal meets the x -axis.

7 Find the coordinates of the turning point on the curve $y = x e^x$ and show whether it is a maximum or minimum point.

- 8 Find the coordinates and nature of any turning points on the curve $y = \frac{e^x}{x}$.
- 9 a Find the stationary point of the curve $y = e^x + e^{-x}$.
b Show that this is a local minimum.
- 10 a Find the coordinates and nature of any turning points on the curve $y = x^2 e^x$.
b Sketch the curve.
- 11 Given that $y = e^x + \frac{1}{e^x}$, show that $y \frac{dy}{dx} = e^{2x} - \frac{1}{e^{2x}}$.
- 12 Given that $y = e^x - e^{-x}$, show that $\left(\frac{dy}{dx}\right)^2 = y^2 + 4$.
- 13 Show that if $y = e^{2x} + e^{-2x}$ then $\frac{d^2y}{dx^2} = 4y$.
- 14 Given that $y = A + B e^{-4x}$, where A and B are constants, show that $\frac{d^2y}{dx^2} + 4 \frac{dy}{dx} = 0$.
- 15 Given that $y = e^{2t}(A + Bt)$, where A and B are constants, show that $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$.
- 16 Find the coordinates of the turning point on the curve $y = 2e^{3x} + 8e^{-3x}$, and determine the nature of this turning point.

4.5 Differentiating $\ln x$

To find $\frac{d}{dx}(\ln x)$ we use the result $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$.

This result is justified below.

Now if $y = \ln x$, [Remember $\ln x = \log_e x$]
then $x = e^y$ [1]

Differentiate with respect to y .

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{e^y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \quad [\text{from [1]}]$$

So if $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$ Remember this result.

Notice that in each case we have $y = \ln[f(x)]$

$$\text{and } \frac{dy}{dx} = \frac{1}{f(x)} \times f'(x).$$

This is a quick way of differentiating $\ln[f(x)]$.

Example 3

Find $\frac{dy}{dx}$ in each case.

a $y = (x^2 + 1) \ln 2x$

We have $y = uv$, where $u = x^2 + 1$ and $v = \ln 2x$.

$$\text{Differentiating, } \frac{du}{dx} = 2x, \frac{dv}{dx} = \frac{1}{2x} \times 2 = \frac{1}{x}.$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= (x^2 + 1) \frac{1}{x} + (\ln 2x) 2x \\ &= \frac{x^2 + 1}{x} + 2x \ln 2x \end{aligned}$$

b $y = e^x \ln x^2$

We have $y = uv$, where $u = e^x$ and $v = \ln x^2$.

$$\text{Differentiating, } \frac{du}{dx} = e^x, \frac{dv}{dx} = \frac{1}{x^2} \times 2x = \frac{2}{x}.$$

$$\begin{aligned} \frac{dy}{dx} &= u \frac{dv}{dx} + v \frac{du}{dx} \\ &= e^x \frac{2}{x} + (\ln x^2) e^x \\ &= \frac{2e^x}{x} + e^x \ln x^2 \end{aligned}$$

EXERCISE 5

1 Find $\frac{dy}{dx}$ for each of the following.

a $y = \ln 4x$

b $y = \ln x^3$

c $y = 6 \ln x$

d $y = \ln(3x - 1)$

e $y = \ln(1 - 2x)$

f $y = \ln(x^3 + x)$

g $y = \ln \frac{x+1}{2}$

h $y = \ln 4x^2$

i $y = 3 \ln(x + 2)$

j $y = \ln \frac{1}{x}$

k $y = \ln \sqrt{x}$

l $y = \ln(x^2 + x - 2)$

2 a Write $\ln \left(\frac{x+4}{x-2} \right)$ as the difference of 2 logarithms.

b Hence find $\frac{d}{dx} \left[\ln \left(\frac{x+4}{x-2} \right) \right]$

3 Differentiate with respect to x .

a $y = \ln\left(\frac{3x}{x+1}\right)$

b $y = \ln\left(\frac{2x+3}{4x-1}\right)$

4 Find $\frac{dy}{dx}$ for each of the following.

a $y = x \ln x$

b $y = x^2 \ln x$

c $y = x \ln(1+x)$

d $y = \frac{\ln x}{x}$

e $y = \frac{\ln x}{(x+1)^2}$

f $y = \frac{x}{\ln x}$

g $y = x^2 + \ln x$

h $y = 3 \ln x + \frac{1}{x}$

i $y = \ln \sqrt{x} + \sqrt{x}$

j $y = \frac{1 + \ln x}{x}$

k $y = x^2 \ln(1+x^2)$

l $y = \ln\left(\frac{x+1}{x+2}\right)$

5 If $y = x^2 \ln x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

6 Given that $y = \ln(\ln x)$, show that $x \ln x \frac{dy}{dx} = 1$.

7 Given that $y = \ln(1+e^x)$, find $\frac{dy}{dx}$ and show that $(1+e^x) \frac{d^2y}{dx^2} = \frac{dy}{dx}$.

8 a Sketch the curves $y = \ln x$ and $y = \ln(x+1)$.

b Find the equation of the tangent to the curve $y = \ln(x+1)$ at the point $(2, \ln 3)$.

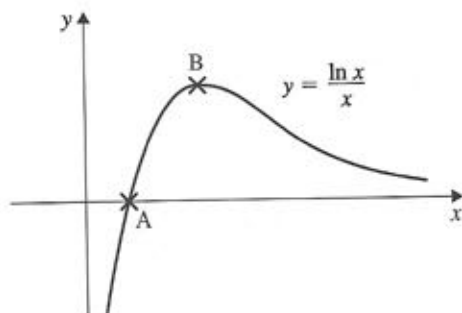
9 Find the equation of the normal to the curve $y = \ln(1+x^2)$ at the point where $x = 1$.

10 a Find the equation of the tangent to the curve $y = \ln(x-1)$ at the point where the curve meets the x -axis.

b The tangent above meets the x and y axes at the points A and B respectively. Find the area of triangle OAB where O is the origin.

11 The diagram shows the curve $y = \frac{\ln x}{x}$.

The curve crosses the x -axis at A and B is the maximum point on the curve. Find the coordinates of A and B.



12 Find the coordinates of the point on the curve $y = x \ln x$ where the gradient is zero.

13 Find the x -coordinate of the point on the curve $y = x \ln 2x$ where the gradient is zero.

EXERCISE 6

1 Differentiate the following with respect to x .

a $\sin x$

b $\cos 3x$

c $4 \tan x$

d $\sin 6x$

e $\cos \frac{3}{2}x$

f $5 \sin 2x$

g $\tan \frac{1}{2}x$

h $10 \cos x + \sin x$

i $\tan x - \cos x$

j $\sin(x+1)$

k $\tan(2x-1)$

l $-\cos \frac{1}{2}x$

2 Find $\frac{dy}{dx}$ in the following.

a $y = (\sin x)^2$

b $y = 3(\cos x)^2$

c $y = \sin^3 x$

d $y = 2 \cos^3 x$

e $y = \sqrt{\cos x}$

f $y = \cos^2 4x$

g $y = \tan^2 9x$

h $y = x + \sqrt{\sin x}$

i $y = (\sin 2x)^{\frac{3}{2}}$

3 Differentiate the following with respect to x .

a $x \sin x$

b $x \cos 2x$

c $x^2 \sin x$

d $\sin x \cos x$

e $\frac{\cos x}{x}$

f $\frac{\sin 2x}{x^2}$

g $\frac{x}{\sin x}$

h $\frac{x^3}{\cos x}$

i $\frac{\cos x}{\sin x}$

4 Given $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, differentiate $\sin\left(\frac{\pi}{2} - x\right)$ to show that $\frac{d}{dx}(\cos x) = -\sin x$.

5 Differentiate $\frac{\sin x}{\cos x}$ to show that $\frac{d}{dx}(\tan x) = \sec^2 x$.

6 Find the gradient of each curve at the point given.

a $y = 3 \sin 2x$ at $\left(\frac{\pi}{2}, 0\right)$.

b $y = 4 \sin x - \cos x$ at $(0, -1)$.

c $y = (\cos 2x)^4$ at $\left(\frac{\pi}{8}, \frac{1}{4}\right)$.

d $y = x \tan x$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

7 Given that $y = \cos 2x$, show that $\frac{d^2y}{dx^2} = -4y$.

8 Given that $y = \sin x + \cos x$, show that $\frac{d^2y}{dx^2} + y = 0$.

9 Find the equation of the tangent to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

10 Differentiate with respect to x .

a $\sin \pi x$

b $\tan 2\pi x$

c $\cos(x - \pi)$

d $\cos \frac{\pi}{2}x$

e $\sin x^2$

f $\tan x^3$

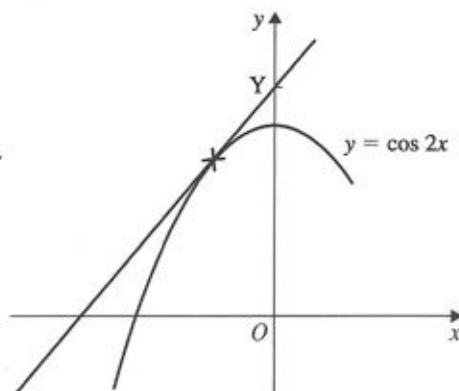
g $\sin(2x - \pi)$

h $\sin \frac{x}{\pi}$

i $\sin^2 x^2$

- 11 Consider the curve $y = \sin x^2$ for $0 \leq x \leq \pi$.
Find the x coordinates of the two points on the curve where the gradient is zero.
- 12 Show that the equation of the tangent to the curve $y = \sin x$ at the point $(\pi, 0)$ is $y + x = \pi$.
- 13 Find the equation of the normal to the curve $y = x - \cos x$ at the point where $x = \frac{\pi}{2}$.
- 14 Show that, at $x = \frac{\pi}{4}$, $\frac{d}{dx}(\sin^2 x + \cos^3 x) = \frac{2\sqrt{2}-3}{2\sqrt{2}}$.
- 15 Find the maximum value of $y = x + \sin 2x$ for $0 < x < \frac{\pi}{2}$.
- 16 Find the minimum value of $y = \tan x - 8 \sin x$ which is given by a value of x between 0 and $\frac{\pi}{2}$.
- 17 Show that for $0 < x < \frac{\pi}{2}$, $f(x) = x \sin x + \cos x$ is an increasing function.
- 18 a If $y = \frac{1 + \cot x}{1 - \cot x}$, show that $\frac{dy}{dx} = \frac{2}{\sin 2x - 1}$.
b Find the gradient of the curve at $x = \frac{\pi}{12}$.

- 19 The tangent to the curve $y = \cos 2x$ at the point where $x = -\frac{\pi}{6}$ meets the y -axis at the point Y .
Find the distance OY where O is the origin.



- 20 Given $f(\theta) = 4 \cos \theta - \sin^2 \theta$, find $f'\left(\frac{\pi}{2}\right)$.
- 21 Find the coordinates of the stationary points on the curve $y = \sin x + \cos x$ for $0 \leq x \leq 2\pi$.
Determine the nature of each point.
- 22 a If $y = \sin x(\cos x)^2$, find $\frac{dy}{dx}$.
b Find the three values of x between 0 and π radians at which $\frac{dy}{dx} = 0$. Give your answers as decimals correct to 2 decimal places.

- 23 a Find the equation of the tangent to the curve $y = \sin x$ at the point where $x = \frac{4\pi}{3}$.

b Find the coordinates of the point where this tangent meets the y -axis.

- 24 a Find the gradient of the curve $y = \sin \pi x$ at the point where $x = 1$.

b Find the equation of the normal to the curve $y = \sin \pi x$ at the point where $x = 1$.

Derivatives of cosec x , cot x and sec x

The derivatives of cosec x , cot x and sec x can be found as follows.

a $y = \operatorname{cosec} x = \frac{1}{\sin x} = (\sin x)^{-1}$

$$\frac{dy}{dx} = -(\sin x)^{-2} \cos x$$

$$= -\frac{\cos x}{\sin^2 x}$$

$$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cot x$$

b $y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1}$

$$\frac{dy}{dx} = -(\cos x)^{-2}(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$

$$= \sec x \tan x$$

c $y = \cot x = (\tan x)^{-1}$

$$\frac{dy}{dx} = -(\tan x)^{-2} \sec^2 x$$

$$= -\frac{\cos^2 x}{\sin^2 x} \times \frac{1}{\cos^2 x}$$

$$= -\operatorname{cosec}^2 x$$

In summary:

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

Some people learn these results, while others, including the authors of this book, can never remember them and work them out each time!

EXERCISE 7

- 1 Differentiate with respect to x .

a cosec x

b cot x

c sec $2x$

d cot $3x$

e cosec $4x$

f $2 \cot x$

g sec $(2x + 1)$

h $x^2 + \sin x + \sec x$

- 2 Find $\frac{dy}{dx}$.

a $y = (\cot x)^2$

b $y = (\sec x)^2$

c $y = \tan x \sec x$

d sec $^2 3x$

e $x \sec x$

f $x^2 \cot 2x$

g $\frac{\sec x}{x}$

h $\frac{x^2}{\cot x}$

i $(1 + \sec x)^2$

- 3 Show that $\frac{d}{dx} \left(\frac{\tan x}{1 + \sec x} \right) = \frac{1}{1 + \cos x}$

4 Given that $y = \cot x$, show that $\frac{d^2y}{dx^2} = 2(1 + y^2)y$.

5 If $y = \sqrt{\sec x + \tan x}$, show that $\frac{dy}{dx} = \frac{1}{2}y \sec x$.

6 Work out $\int \frac{2 + \cos x}{\sin^2 x} dx$.

4.7 Further questions on differentiation

Here is a summary of the results used in this section.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \quad (\text{chain rule})$$

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x \quad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

In this section you will do questions where more than one of these results may be required.

Example 1

Find $\frac{dy}{dx}$.

a $y = e^x \sin x^2$

$$\begin{aligned} \frac{dy}{dx} &= e^x(\cos x^2)2x + \sin x^2 e^x \quad (\text{product rule}) \\ &= e^x(2x \cos x^2 + \sin x^2) \end{aligned}$$

b $y = e^{2x} \ln(1 + x)$

$$\begin{aligned} \frac{dy}{dx} &= e^{2x} \times \frac{1}{(1+x)} + \ln(1+x) \times 2e^{2x} \quad (\text{product rule}) \\ &= e^{2x} \left[\frac{1}{1+x} + 2 \ln(1+x) \right] \end{aligned}$$

c $y = \frac{\sin 4x}{e^x}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^x \times 4 \cos 4x - \sin 4x \times e^x}{(e^x)^2} \quad (\text{quotient rule}) \\ &= \frac{4 \cos 4x - \sin 4x}{e^x} \end{aligned}$$

$$\text{From [2]} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{y}} = y \quad \therefore \quad \frac{dy}{dx} = \frac{1}{5}e^x$$

EXERCISE 8

1 Differentiate the following:

a $\ln x$

b $\ln(2x)$

c $2 \ln(3x)$

d $x + \ln(2x)$

e $5e^x + \ln(3x)$

f $2e^x + 5 \ln x$

2 a Find the exact coordinates of the stationary point of the curve $y = \ln x - 3x + 1$.

b Show that this is a local maximum.

3 a Find the exact coordinates of the stationary point of the curve $y = \ln(4x) - x - 2$.

b Show that this is a local maximum.

4 a Show that the curve $y = \ln(2x) + x^2$ passes through $(\frac{1}{2}, \frac{1}{4})$.

b Find the equation of the normal to $y = \ln(2x) + x^2$, at $(\frac{1}{2}, \frac{1}{4})$.

5 Find the equation of the tangent to the curve $y = \ln x + x^2 + 3x$ at the point $(1, 4)$.

6 Find $\frac{dy}{dx}$ in the following:

a $y = 5e^x$

b $y = 7e^x$

c $y = 5e^{4x}$

d $y = \ln(x + 1)$

e $y = \frac{4}{e^x}$

f $y = (e^{3x} + 1)^2$

g $y = 2e^x + \ln 3x$

h $y = \frac{5e^{3x} + 2e^{2x}}{e^{2x}}$

i $y = \frac{e^x + 1}{e^x - 1}$

7 If $y = (e^{\frac{x}{2}} + 1)(e^{\frac{x}{2}} - 1)$ then find $\frac{dy}{dx}$.

8 Differentiate the following:

a $y = \ln(x^2)$

b $y = \ln(2x^3)$

c $y = \ln(\sqrt{x})$

d $y = \ln\left(\frac{1}{x}\right)$

e $y = \ln(\sqrt{9x})$

f $y = \ln\left(\frac{1}{\sqrt[3]{x}}\right)$

9 Find $\frac{dy}{dx}$ in the following.

a $y = (x + 1)^2$

b $y = (2x + 1)^7$

c $y = (3x - 5)^8$

d $y = 5(3x - 7)^3$

e $y = (4x + 3)^{-3}$

f $y = \frac{1}{(5x - 2)^4}$

g $y = (4x + 11)^{\frac{1}{2}}$

h $y = \sqrt[3]{(15x - 17)}$

i $y = (ax + b)^n$

10 Show that the equation of the tangent to the curve $y = 2e^x$ at the point where $x = \frac{1}{2}$ is $y = \sqrt{e}(2x + 1)$.

- 11 Find the turning point of the curve $y = x^2 - 2\ln x$, where $x > 0$, and show that this is a local minimum.
- 12 Find the turning point of the curve $y = x - e^x$ and show that this is a local maximum.
- 13 Find $\frac{dy}{dx}$ in the following.
- | | | |
|------------------------------------|--------------------|-----------------------|
| a $y = x \ln x$ | b $y = x^2 e^x$ | c $y = \sin x \cos x$ |
| d $y = \sqrt{x} e^x$ | e $y = e^x \tan x$ | f $y = x \cot x$ |
| g $y = e^x \operatorname{cosec} x$ | h $y = x^3 \sec x$ | i $y = x^3 \cos x$ |
- 14 If $y = x^n e^x$ then show that $\frac{dy}{dx} = x^{n-1} e^x (x + n)$.
- 15 Find $\frac{dy}{dx}$ in the following, simplifying and factorising where possible:
- | | | |
|-------------------------------|-------------------------------|----------------------------|
| a $y = \frac{\ln x}{x}$ | b $y = \frac{e^x}{x^2}$ | c $y = \frac{\sec x}{e^x}$ |
| d $y = \frac{\sin x}{\cos x}$ | e $y = \frac{\cos x}{\sin x}$ | f $y = \frac{x^2}{e^x}$ |
| g $y = \frac{x+1}{\sqrt{x}}$ | h $y = \frac{x^3}{\cos x}$ | i $y = \frac{\cot x}{x^2}$ |
- 16 a If $y = \frac{2x+3}{\sqrt{x}}$ then show that $\frac{dy}{dx} = \frac{2x-3}{2x\sqrt{x}}$.
 b Show that the turning point is at $(\frac{3}{2}, 2\sqrt{6})$.
- 17 The gradient of the curve $y = e^x(ax^2 + bx + c)$ is zero at $x = 1$ and at $x = -2$. The curve passes through the point $(0, -1)$. Find the values of a , b and c .
- 18 Find $\frac{dy}{dx}$ in the following.
- | | | |
|---------------------|----------------------|---|
| a $y = \sin 2x$ | b $y = e^{5x}$ | c $y = e^{\sin x}$ |
| d $y = \cos(x^2)$ | e $y = e^{\sec x}$ | f $y = \operatorname{cosec}\left(3x + \frac{\pi}{3}\right)$ |
| g $y = \cot(3x)$ | h $y = \ln(x^3 + 3)$ | i $y = \ln(\sin 2x)$ |
| j $y = \ln(\cos x)$ | k $y = e^{x^3}$ | l $y = \ln(\sec x)$ |
| m $y = (x^2 + 1)^3$ | n $y = \sin^3 x$ | o $y = \tan^3(2x)$ |
- 19 If $x = \cos y$, find $\frac{dy}{dx}$. Give your answer in terms of x .
- 20 If $x = \sin 2y$, find $\frac{dy}{dx}$ in terms of x .

21 Find $\frac{dy}{dx}$, in terms of x .

a $x = \tan y$

b $x = \tan 3y$

c $x = e^{2y}$

d $x = e^{-y}$

e $x = \ln 5y$

f $x = \ln y^2$

g $x = \sin\left(2y - \frac{\pi}{6}\right)$

22 a Consider $x = y^2$, $y \geq 0$. Find $\frac{dy}{dx}$, using $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

b Write $y = \sqrt{x}$ and find $\frac{dy}{dx}$. Confirm that your result is the same as that obtained in part a.

23 Express $\frac{dy}{dx}$ in terms of $f(x)$ and $f'(x)$:

a $y = f(x)^n$

b $y = e^{f(x)}$

c $y = \sin(f(x))$

24 a If $\log_a x = p$, $\log_e a = q$ and $\log_e x = r$ then write down three equations involving p , q and r without logarithms.

b Hence show that $p = \frac{r}{q}$.

c Hence express $\log_a x$ in terms of $\ln x$ and $\ln a$ and so show that if $y = \log_a x$ where a is a constant then $\frac{dy}{dx} = \frac{1}{x \ln a}$.

REVIEW EXERCISE 4

1 Differentiate the following with respect to x .

a $y = (3x - 1)^4$

b $y = 5(3x - 4)^3$

c $y = (4x + 3)^{-2}$

d $y = \frac{1}{(5x - 1)^3}$

e $y = \frac{1}{(2 - 3x)^4}$

f $y = (3x + 1)^{\frac{1}{2}}$

2 Differentiate the following with respect to x .

a $y = (x^2 + 3)^4$

b $y = (2 - x^3)^5$

c $y = (3x^2 + 1)^5$

d $y = (x^2 + x)^4$

e $y = \left(1 + \frac{1}{x}\right)^3$

f $y = \left(x^2 + \frac{1}{x}\right)^{-3}$

3 Differentiate with respect to x , simplify your results.

a $y = x(x + 1)^3$

b $y = (x + 1)(2x - 3)^4$

c $y = (x + 1)^3(x - 1)^2$

d $y = x^2(3x - 2)^2$

- 4 Sketch the curve $y = (x - 1)^2(x + 1)$, showing the coordinates of
- the points where it meets the axes
 - the turning points.
- 5 Find the coordinates of the turning points of the curve $y = (x - 2)(x + 3)^2$. Determine the nature of each turning point.
- 6 Differentiate with respect to x .
- $\frac{x^2}{x + 1}$
 - $\frac{\sqrt{x + 1}}{x}$
 - $\frac{x + 2}{x + 1}$
 - $\frac{e^{3x}}{x^2}$
 - $\frac{3 - 2x}{x^2}$
 - $\frac{\sin x}{x}$
- 7 Show that the curve $y = \frac{x^3}{(1 + x^4)^{\frac{1}{2}}}$ has a positive gradient for all values of x except at $x = 0$.
- 8 Differentiate the following with respect to x .
- $y = e^x$
 - $y = -3e^x$
 - $y = \frac{1}{2}e^x$
 - $y = 2x^3 - 4e^x$
 - $y = 3(e^x - x^{\frac{1}{2}})$
 - $y = \frac{1}{2}(x^{\frac{1}{2}} - \frac{1}{2}e^x)$
- 9 Differentiate the following with respect to x .
- $y = \ln x$
 - $y = 4 \ln x$
 - $y = \ln 2x$
 - $y = \ln x^2$
 - $y = \ln x^{\frac{1}{2}}$
 - $y = \ln\left(\frac{x}{2}\right)$
 - $y = \ln\left(\frac{1}{x}\right)$
- 10 Find the gradient of the following curves at the given points.
- $y = e^x - x$: $x = 2$
 - $y = 3e^x - 2x$: $x = 0$
 - $y = \ln 2x - \sqrt{x}$: $x = 4$
 - $y = x^3 - 3 \ln x^2$: $x = 2$
- 11 Find the equation of the tangent to the following curves at the given point.
- $y = e^x$: $x = 0$
 - $y = \ln x$: $x = 1$
 - $y = 1 - e^x$: $x = 2$

12 Find the equation of the normal to the following curves at the given point.

a $y = e^x$: $x = 1$

b $y = \ln x^2$: $x = 3$

13 Find the coordinates of the stationary point of the following curves and show whether it is a maximum or a minimum.

a $y = e^x - x$

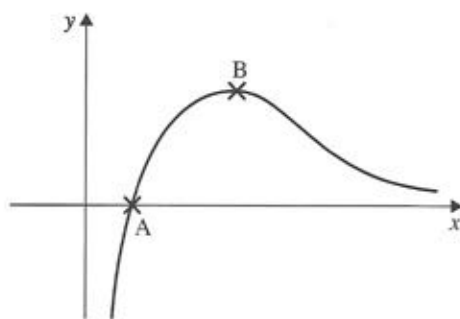
b $y = e^x + e^{-x}$

c $y = -x + \ln x$

d $y = 2x^2 - \ln x$

e $y = \ln x - 8x + 1$

14 The curve $y = \frac{\ln x}{x}$ cuts the x -axis at A and has a maximum point at B. Find the coordinates of A and B.



15 If $y = (e^x + 1)^2$, show that $\frac{dy}{dx} = 2e^x(e^x + 1)$.

16 Differentiate the following with respect to x .

a $\sin x + 3$

b $2 \cos x$

c $\sin x + \cos x$

d $2 \sin x - 3 \cos x$

e $\tan x + x^2 + 4$

f $3 \sin x - \tan x$

17 Find the gradient of the following curves at the given point.

a $y = \sin x$: $x = \frac{\pi}{3}$

b $y = \cos x$: $x = \frac{\pi}{2}$

c $y = \cos x + \sin x$: $x = \frac{\pi}{4}$

d $y = x + \cos x$: $x = \frac{\pi}{6}$

e $y = 3 - \cos 2x$: $x = \frac{\pi}{12}$

18 Find the equation of the tangent to the curve $y = x - \cos x$ where $x = \frac{\pi}{2}$

19 Find the equation of the normal to the curve $y = 2 \sin x + \cos x$ where $x = \frac{\pi}{2}$

20 Differentiate the following with respect to x .

a $y = \sin 3x$

b $y = \cos 4x$

c $y = \tan 5x$

d $y = \sin^2 x$

e $y = \tan^4 x$

f $y = \sin\left(2x + \frac{\pi}{4}\right)$

g $y = \sqrt{\sin x}$

h $y = \sin^2 3x$

i $y = \cos^3 2x$

21 Differentiate the following with respect to x .

a $y = e^{2x}$

b $y = e^{-3x}$

c $y = e^{x^2}$

d $y = \frac{1}{e^x}$

e $y = e^{\sin x}$

f $y = \ln(x^2 + 1)$

g $y = \ln(x^3 - 2)$

h $y = \ln(x^3 + 2x - 1)$

i $y = \ln\left(\frac{x^2 + 1}{x - 1}\right)$

j $y = \ln\left(\frac{1}{x^2 + 3}\right)$

k $y = e^{\ln x}$

l $y = e^{\sqrt{x}}$

22 Show that if $y = (e^x + e^{-x})^2$ then $\frac{dy}{dx} = 2(e^{2x} - e^{-2x})$.

23 Given that $y = e^x \ln(1 + \sin x)$, find $\frac{dy}{dx}$ in terms of x .

24 Differentiate with respect to x .

a $y = x e^x$

b $y = x \ln x$

c $y = e^x \ln x$

d $y = x \sin x$

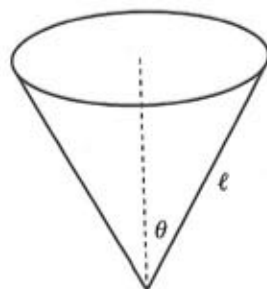
e $y = x^2 \cos x$

f $y = e^x \tan x$

25 The diagram shows a right circular cone of slant height ℓ .

a Show that the volume V of the cone is given by $V = \frac{1}{3} \pi \ell^3 \sin^2 \theta \cos \theta$

b Given that ℓ is constant and that θ varies, find the maximum value of V in terms of ℓ and π .



26 Find the coordinates of the points on the curve $y = \sin x \cos^3 x$ where the gradient is zero. Work in the range $0 < x < \pi$.

27 Differentiate the following with respect to x .

a $y = \sin^2 x \cos x$

b $y = \ln(2x + 5)$

c $y = \sin 2x \cos 3x$

e $y = (\ln x)^2$

f $y = \sin x^2$

d $y = e^{x^2+2}$

28 Find $\frac{dy}{dx}$.

a $y = \sec x$

b $y = \operatorname{cosec} x$

c $y = \cot x$

29 Find the gradient of the curve $y = \sec 3x$ at the point where $x = \frac{\pi}{9}$.

- 30 Find the coordinates of the stationary point on the curve $y = 2x - \ln(x^4)$ and determine the nature of this stationary point.
- 31 Use $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ to find $\frac{dy}{dx}$ in terms of x for the following.
- a $x = \sin y$ b $x = e^{3y}$ c $x = \ln y$ d $x = \ln 2y$
- 32 Given that $x = y^3 \ln 2y$, $y > 0$
- a Find $\frac{dx}{dy}$
- b Use your answer to part (a) to find, in terms of e , the value of $\frac{dy}{dx}$ at $y = e$.
- 33 Given that $y = \frac{e^x}{x^2 - 3}$, find the x coordinates of the two points on the curve where the gradient is zero.

EXAMINATION EXERCISE 4

- 1 Differentiate with respect to x
- i $(3 - 2x)^5$,
- ii $\ln(4x + 7)$. [OCR]
- 2 Find $\frac{dy}{dx}$ for each of the following cases
- a $y = e^{2x} \sin 3x$,
- b $y = (2x^2 + 1)^5$. [AQA]
- 3 Find the equation of the tangent to the curve $y = (3x - 4)^5$ at the point for which $x = 2$, giving your answer in the form $y = mx + c$. [OCR]
- 4 a Differentiate $\sqrt{1 + x^2}$.
- b Differentiate $\frac{1 + x}{1 - x}$. [MEI]
- 5 a Differentiate:
- i $2x^{\frac{1}{2}}$;
- ii $\ln(x + 1)$;
- b Hence show that $\int_1^4 \left(x^{-\frac{1}{2}} + \frac{1}{x+1} \right) dx = 2 + \ln \frac{5}{2}$. [AQA]
- 6 The equation of a curve is $y = xe^{-2x}$.
Find the coordinates of the turning point on the curve. [OCR]

- 7 Fig. 1 shows the graph of $y = x\sqrt{1+x}$. The point P on the curve is on the x-axis.

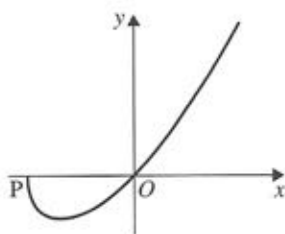


Fig. 1

- Write down the coordinates of P.
 - Show that $\frac{dy}{dx} = \frac{3x+2}{2\sqrt{1+x}}$.
 - Hence find the coordinates of the turning point on the curve.
What can you say about the gradient of the curve at P? [MEI]
- 8 The curve C has equation $y = 2e^x + 3x^2 + 2$. The point A with coordinates (0, 4) lies on C. Find the equation of the tangent to C at A. [EDEXCEL]
- 9 Use the derivatives of $\sin x$ and $\cos x$ to prove that the derivative of $\tan x$ is $\sec^2 x$. [AQA]
- 10 Find the equation of the tangent to the curve $y = \frac{2+x}{\cos x}$ at the point on the curve where $x = 0$. [AQA]

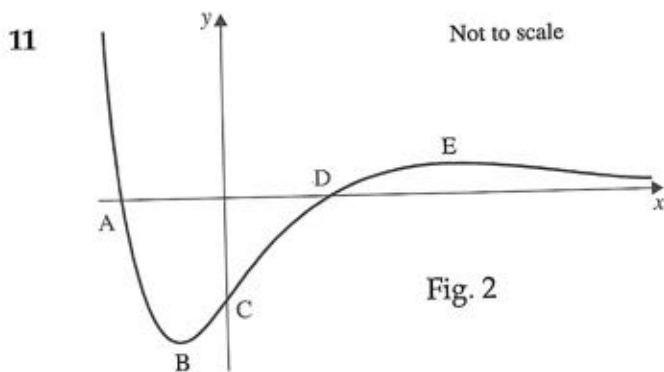


Fig. 2

Fig. 2 shows a sketch of the graph of $y = (x^2 - 3)e^{-x}$. The graph crosses the x-axis at points A and D and the y-axis at C. Points B and E are stationary points on the curve.

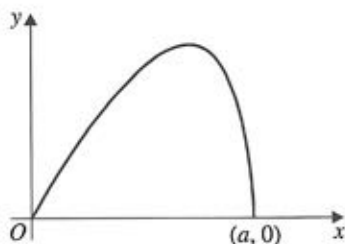
- Find the coordinates of the points A, C and D.
- Show that $\frac{dy}{dx} = -(x^2 - 2x - 3)e^{-x}$.
- Deduce that the x-coordinates of the points B and E are -1 and 3 respectively, and find the corresponding y-coordinates. [MEI]

12 The function f is defined for all $x > 0$ by $f(x) = 4 \ln x - \frac{1}{x}$.

- a Find the derivative $f'(x)$.
- b Explain why f is an increasing function.
- c State, giving a reason, whether f^{-1} exists.

[AQA]

13 Fig. 1 shows part of the graph with equation $y = x\sqrt{9 - 2x^2}$. It crosses the x -axis at $(a, 0)$.



- i Find the value of a , giving your answer as a multiple of $\sqrt{2}$.
- ii Show that the result of differentiating $\sqrt{9 - 2x^2}$ is $\frac{-2x}{\sqrt{9 - 2x^2}}$.

Hence show that if $y = x\sqrt{9 - 2x^2}$ then

$$\frac{dy}{dx} = \frac{9 - 4x^2}{\sqrt{9 - 2x^2}}.$$

- iii Find the x -coordinate of the maximum point on the graph of $y = x\sqrt{9 - 2x^2}$. Write down the gradient of the curve at the origin. What can you say about the gradient at the point $(a, 0)$?

[MEI]

14 A curve has equation $y = (x^2 + 5x + 4) \cos 3x$.

- a Find $\frac{dy}{dx}$.
- b Find the equation of the tangent to the curve at the point where $x = 0$.

[AQA]

15 A curve has equation

$$y = \frac{2x}{\sin x}, \quad 0 < x < \pi.$$

- a Find $\frac{dy}{dx}$.
- b The point P on the curve has coordinates $\left(\frac{\pi}{2}, \pi\right)$.
 - i Show that the equation of the tangent to the curve at P is $y = 2x$.
 - ii Find the equation of the normal to the curve at P , giving your answer in the form $y = mx + c$.

[AQA]

16 It is given that

$$y = \ln x - 2x^2 + 3, \quad x > 0$$

- a Find $\frac{dy}{dx}$.
- b Verify that y has a stationary value when $x = \frac{1}{2}$.
- c Find the value of $\frac{d^2y}{dx^2}$ when $x = \frac{1}{2}$.
- d Hence determine whether this stationary value is a maximum or a minimum.

[AQA]

17 A curve has equation

$$y = x^2 - 3x + \ln x + 2, \quad x > 0.$$

- a i Find $\frac{dy}{dx}$.
- ii Hence show that the gradient of the curve at the point where $x = 2$ is $\frac{3}{2}$.
- b i Show that the x -coordinates of the stationary points of the curve satisfy the equation

$$2x^2 - 3x + 1 = 0$$

- ii Hence find the x -coordinates of each of the stationary points.

iii Find $\frac{d^2y}{dx^2}$.

- iv Find the value of $\frac{d^2y}{dx^2}$ at each of the stationary points.

- v Hence show that the y -coordinate of the maximum point is

$$\frac{3}{4} - \ln 2.$$

[AQA]

18 A curve has equation

$$y = x^2 - \ln x, \quad x > 0$$

The curve has one stationary point P .

- a Show that the x -coordinate of P is $\frac{1}{\sqrt{2}}$.
- b Give the y -coordinate of P in the form $a + b \ln 2$, where a and b are rational numbers.
- c Find the value of $\frac{d^2y}{dx^2}$ at P .
- d State, giving a reason, whether P is a maximum or a minimum point on the curve.

[AQA]

- 19 The function f is defined for $x > 0$ by

$$f(x) = e^{-2x} + \frac{3}{x} + 3$$

- a i Differentiate $f(x)$ with respect to x to find $f'(x)$.
 ii Hence prove that f is a decreasing function.
 b Find the range of f .

[AQA]

- 20 A curve has equation $y = e^{3x} - 24x$.

- a Show that the curve cuts the x -axis at the point where $x = \alpha$, with $1 < \alpha < \ln 3$.
 b Determine $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of x .
 c The curve has a single turning point, P . Find the x -coordinate of P in an exact form, and show that its y -coordinate is $8(1 - 3 \ln 2)$. Find the value of $\frac{d^2y}{dx^2}$ at P and hence deduce the nature of the turning point.

21

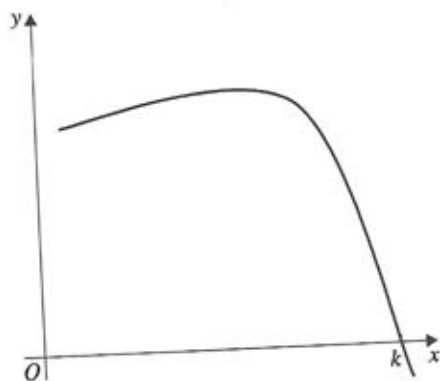


Figure 1 shows a sketch of the curve with equation $y = f(x)$, where

$$f(x) = 10 + \ln(3x) - \frac{1}{2}e^x, \quad 0.1 \leq x \leq 3.3.$$

Given that $f(k) = 0$,

- a Show, by calculation, that $3.1 < k < 3.2$.
 b Find $f'(x)$.

The tangent to the graph at $x = 1$ intersects the y -axis at the point P .

- c i Find an equation of this tangent.
 ii Find the exact y -coordinate of P , giving your answer in the form $a + \ln b$.

[EDEXCEL]